ATRISCAL coordinate estimation by steepest descent method

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In Shojima (2009), the stress function of asymmetric triangulation scaling (ATRISCAL) is proposed to be given by the following expression:

$$F(\boldsymbol{X}) = \sum_{c,r\ (r\neq c)}^{n+1} \lambda_{c|r}\Big(p_{c|r} - \frac{|\overrightarrow{OX_{rc}}|}{|\overrightarrow{OX_r}|}\Big)^2 = \sum_{c,r\ (r\neq c)}^{n+1} \lambda_{c|r}\Big(p_{c|r} - \frac{|\boldsymbol{x}_{rc}|}{|\boldsymbol{x}_r|}\Big)^2, \tag{1}$$

where n is the number of items. In addition, $p_{c|r}$ is the conditional correct response rate of item c when item r is answered correctly, that is, $p_{c|r} = p_{rc}/p_r$, where p_r is the correct response rate of item r and p_{rc}, the joint correct response rate of items r and c. In addition, item $n+1$ is an imaginary item with a correct response rate of 1.0, that is, $p_{n+1} = 1.0$. Therefore, $p_{n+1|r} = 1.0$ and $p_{c|n+1} = p_c$.

In addition, $\boldsymbol{X} = \{x_{rm}\}$ $((n+1)\times M)$ gives the M-dimensional coordinates of the $n+1$ items to be estimated, where $\boldsymbol{x}_r$ $(M\times 1)$ is the r-th row vector in $\boldsymbol{X}$. Furthermore, $\lambda_{c|r}$ is a nonnegative weight and usually, $\lambda_{c|r} = \lambda_{r|c}$. Furthermore,

$$\overrightarrow{OX_{rc}} = -\frac{(\overrightarrow{OX_r}\cdot\overrightarrow{X_rX_c})\overrightarrow{OX_c} - (\overrightarrow{OX_c}\cdot\overrightarrow{X_rX_c})\overrightarrow{OX_r}}{|\overrightarrow{X_rX_c}|^2} = -\frac{\{\boldsymbol{x}_r'(\boldsymbol{x}_c-\boldsymbol{x}_r)\}\boldsymbol{x}_c - \{\boldsymbol{x}_c'(\boldsymbol{x}_c-\boldsymbol{x}_r)\}\boldsymbol{x}_r}{|\boldsymbol{x}_c-\boldsymbol{x}_r|^2} = \boldsymbol{x}_{rc} \tag{2}$$

in Equation (1) is the perpendicular foot from the origin O on the line segment X_rX_c. That is,

$$\frac{|\overrightarrow{OX_{rc}}|}{|\overrightarrow{OX_r}|} = \frac{\sqrt{|\overrightarrow{OX_r}|^2|\overrightarrow{OX_c}|^2 - \overrightarrow{OX_r}\cdot\overrightarrow{OX_c}}}{|\overrightarrow{OX_r}||\overrightarrow{X_rX_c}|} = \frac{\sqrt{|\boldsymbol{x}_r|^2|\boldsymbol{x}_c|^2 - (\boldsymbol{x}_r'\boldsymbol{x}_c)^2}}{|\boldsymbol{x}_r||\boldsymbol{x}_c-\boldsymbol{x}_r|} = \pi_{c|r}. \tag{3}$$

Let the number of dimensions be 3 ($M = 3$), and the z-coordinate of each item in the 3D space is constrained to be nonnegative. In other words, $x_{r3} \geq 0$ $(r = 1,\cdots,n)$. In addition, the coordinates of some items in the 3D space are fixed because of spatial indeterminacy. First, the coordinates of the imaginary $n+1$-th item are set to $\boldsymbol{x}_{n+1} = [0\ 0\ 1]'$. Next, for the item with index k whose correct response rate is the lowest, the x- and y-coordinates are set as 0 and a positive value, respectively, that is, $\boldsymbol{x}_k = [0\ x_{k2}(>0)\ x_{k3}]'$. Finally, for the item with index l whose correct response rate conditioned by item k is the median among $p(\cdot|k)$s, the x-coordinate is set to a positive value, that is, $\boldsymbol{x}_l = [x_{l1}(>0)\ x_{l2}\ x_{l3}]'$.

Although the stress function of Equation (1) is straightforward and simple, it tends to produce a degenerate solution. Therefore, using a penalty function $T(\boldsymbol{X})$ against degeneration, the stress function is reconstructed by

$$F^*(\boldsymbol{X}) = \frac{F(\boldsymbol{X})}{T(\boldsymbol{X})}, \tag{4}$$

where

$$T(\boldsymbol{X}) = \sum_{c,r\ (r\neq c)}^{n+1} \delta_{c|r}\lambda_{c|r}\big(\pi_{c|r} - \bar{p}\big)^2. \tag{5}$$

The constant $\delta_{c|r}$ is dichotomous and is coded 1 when the perpendicular foot $\boldsymbol{x}_{rc}$ is located within the line segment between $\boldsymbol{x}_r$ and $\boldsymbol{x}_c$. On the other hand, the constant is coded 0 if the perpendicular foot is located on the extension of the line segment. In addition, $\bar{p}$ in Equation (5) is

$$\bar{p} = \frac{\sum_{c,r\ (r\neq c)}^{n+1} p_{c|r}}{n(n+1)}. \tag{6}$$

To estimate $\boldsymbol{X}$ by minimizing the stress function using the steepest descent method, the first derivatives of the stress function are required. First, the derivative of Equation (4) with respect to $\boldsymbol{x}_j$ $(j = 1, \cdots, n)$ is given by

$$\frac{\partial F^*(\boldsymbol{X})}{\partial \boldsymbol{x}_j} = \frac{1}{T(\boldsymbol{X})}\frac{\partial F(\boldsymbol{X})}{\partial \boldsymbol{x}_j} - \frac{F(\boldsymbol{X})}{\{T(\boldsymbol{X})\}^2}\frac{\partial T(\boldsymbol{X})}{\partial \boldsymbol{x}_j}, \tag{7}$$

where

$$\frac{\partial F(\boldsymbol{X})}{\partial \boldsymbol{x}_j} = \frac{\partial}{\partial \boldsymbol{x}_j}\Big\{\sum_{r\ (\neq j)}^{n+1} \lambda_{j|r}(p_{j|r} - \pi_{j|r})^2 + \sum_{c\ (\neq j)}^{n+1} \lambda_{c|j}(p_{c|j} - \pi_{c|j})^2\Big\} \tag{8}$$

and

$$\frac{\partial T(\boldsymbol{X})}{\partial \boldsymbol{x}_j} = \frac{\partial}{\partial \boldsymbol{x}_j}\Big\{\sum_{r\ (\neq j)}^{n+1} \delta_{j|r}\lambda_{j|r}(\pi_{j|r} - \bar{p})^2 + \sum_{c\ (\neq j)}^{n+1} \delta_{c|j}\lambda_{c|j}(\pi_{c|j} - \bar{p})^2\Big\}. \tag{9}$$

The kernels of the above equations are

$$\frac{\partial(p_{j|r} - \pi_{j|r})^2}{\partial \boldsymbol{x}_j} = -\frac{2\lambda_{j|r}(p_{j|r} - \pi_{j|r})}{\pi_{j|r}|\boldsymbol{x}_r|^2|\boldsymbol{x}_j - \boldsymbol{x}_r|^2}\Big[|\boldsymbol{x}_r|^2(1 - \pi_{j|r}^2)\boldsymbol{x}_j - (\boldsymbol{x}_j'\boldsymbol{x}_r - \pi_{j|r}^2|\boldsymbol{x}_r|^2)\boldsymbol{x}_r\Big], \tag{10}$$

$$\frac{\partial(p_{c|j} - \pi_{c|j})^2}{\partial \boldsymbol{x}_j} = -\frac{2\lambda_{c|j}(p_{c|j} - \pi_{c|j})}{\pi_{c|j}|\boldsymbol{x}_j|^2|\boldsymbol{x}_c - \boldsymbol{x}_j|^2}\Big[\Big\{|\boldsymbol{x}_c|^2 - \pi_{c|j}^2\big(|\boldsymbol{x}_c - \boldsymbol{x}_j|^2 + |\boldsymbol{x}_j|^2\big)\Big\}\boldsymbol{x}_j - \big(\boldsymbol{x}_c'\boldsymbol{x}_j - \pi_{c|j}^2|\boldsymbol{x}_j|^2\big)\boldsymbol{x}_c\Big], \tag{11}$$

$$\frac{\partial(\pi_{j|r} - \bar{p})^2}{\partial \boldsymbol{x}_j} = \frac{2\delta_{j|r}\lambda_{j|r}(\pi_{j|r} - \bar{p})}{\pi_{j|r}|\boldsymbol{x}_r|^2|\boldsymbol{x}_j - \boldsymbol{x}_r|^2}\Big[|\boldsymbol{x}_r|^2(1 - \pi_{j|r}^2)\boldsymbol{x}_j - (\boldsymbol{x}_j'\boldsymbol{x}_r - \pi_{j|r}^2|\boldsymbol{x}_r|^2)\boldsymbol{x}_r\Big], \tag{12}$$

and

$$\frac{\partial(\pi_{c|j} - \bar{p})^2}{\partial \boldsymbol{x}_j} = \frac{2\delta_{c|j}\lambda_{c|j}(\pi_{c|j} - \bar{p})}{\pi_{c|j}|\boldsymbol{x}_j|^2|\boldsymbol{x}_c - \boldsymbol{x}_j|^2}\Big[\Big\{|\boldsymbol{x}_c|^2 - \pi_{c|j}^2\big(|\boldsymbol{x}_c - \boldsymbol{x}_j|^2 + |\boldsymbol{x}_j|^2\big)\Big\}\boldsymbol{x}_j - \big(\boldsymbol{x}_c'\boldsymbol{x}_j - \pi_{c|j}^2|\boldsymbol{x}_j|^2\big)\boldsymbol{x}_c\Big]. \tag{13}$$

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