Asymmetric Triangulation Scaling

A Multidimensional Scaling for Visualizing Inter-Item Dependency Structure

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1. Introduction

It is important to explore the inter-item dependency structure underlying test data. If the structure can be clarified, teachers can better order their study programs and learning materials. Ueno (2002) developed a model combining item response theory and a Bayesian network model to explore inter-item dependency relationships.

We propose a multidimensional scaling (MDS) that visualizes the inter-item dependency structure in an adequate space by analyzing the conditional probability matrix. Let us assume that $P(i)$ is the correct response rate of item i and $P(i,j)$ is the joint correct response rate of items i and j. Then, the conditional probability of correctly answering item j given that item i has been correctly responded to, $P(j|i)$, is $P(j|i)=P(i,j)/P(i)$. A large $P(j|i)$ means that the probability of correctly answering item j increases if item i has been correctly answered. Then, it is highly likely that the knowledge required for item i is a conditional precedent for correctly responding to item j.

2. Method

2.1 Inter-Item Asymmetry Conditional Probability Matrix

The inter-item asymmetry conditional probability (ACP) matrix is an $n \times n$ matrix in which the ij-th element is $P(j|i)$ and n is the number of items. The matrix is asymmetric because $P(j|i)$ is not generally equal to $P(i|j)$. The i-th diagonal entry of the matrix is 1.0 ($P(i|i)= P(i,i)/P(i)= P(i)/P(i)=1.0$).

Next, the imaginary n+1-th item is prepared. The correct response rate of the imaginary item is 1.0. Therefore, the $(n+1,i)$-th element of the expanded ACP matrix becomes $P(i|n+1)=P(i,n+1)/P(n+1)=P(i)$. Moreover, the $(i,n+1)$-th element of the expanded matrix is $P(n+1|i)=P(i,n+1)/P(i)=1.0$, and the $(n+1,n+1)$-th element is 1.0. This expanded ACP matrix is our target of analysis.

2.2 Measure

It is important to determine what property between items i and j in a multidimensional space is measured and recorded as the ij-th element of the expanded ACP matrix. Let us suppose that items i and j are located in a $Q(<n)$-dimensional space with coordinates $\overrightarrow{OX_i} = \boldsymbol{x}_i = [x_{i1} \cdots x_{iQ}]'$ and $\overrightarrow{OX_j} = \boldsymbol{x}_j = [x_{j1} \cdots x_{jQ}]'$. In addition, let us assume that the correct response rates of the two items satisfy $P(i) = |\overrightarrow{OX_i}|$ and $P(j) = |\overrightarrow{OX_j}|$ and the joint correct response rate of the two items satisfy $P(i,j) = |\overrightarrow{OX_{ij}}|$, where $\overrightarrow{OX_{ij}} = \boldsymbol{x}_{ij} = [x_{ij1} \cdots x_{ijQ}]'$ is the perpendicular dropped on line

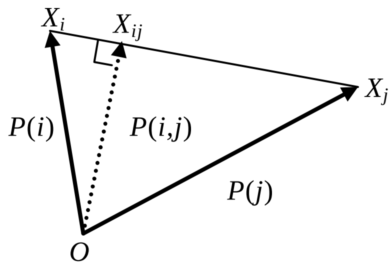

Fig. 1: Relationships among X_i, X_j, and X_{ij}

segment $\overrightarrow{X_iX_j}$ ($\overrightarrow{X_iX_j} \perp \overrightarrow{OX_{ij}}$). Figure 1 shows the relationships among $\overrightarrow{OX_i}$, $\overrightarrow{OX_j}$, and $\overrightarrow{OX_{ij}}$.

From Figure 1, the conditional probabilities $P(j|i)$ and $P(i|j)$ can be regarded as the cosines of angles X_iOX_{ij} and X_jOX_{ij}, respectively. In addition, the cosine of the identical vector being the cosine of angle 0 (cos(0)=1.0), which is consistent with the fact that the diagonal elements of the expanded ACP matrix are 1.0.

Asymmetric TRIangulation SCALing (ATRISCAL) proposed in this study regards the relationships of all item pairs as the cosines of triangles and has the object to restore the lost Q-dimensional coordinates of all n items from the expanded ACP matrix. That is, ATRISCAL seeks the Q-dimensional coordinates of the n items that as much as possible satisfy the asymmetric dependency relationships between all item pairs. Although an (n+1)-dimensional space must be prepared to obtain a result that perfectly explains all $n(n+1)$ nondiagonal entries in the expanded ACP matrix, the number of dimensions is at most three for visualization.

2.3 Stress Function

Let us assume that the lost coordinate matrix to be restored is $\boldsymbol{X}=\{x_{iq}\}$ ($n \times Q$), and the stress function to be minimized is

$$F(\boldsymbol{X}) = \sum_{i,j\ (i \neq j)}^{n+1} \lambda_{ij} \left(P(j|i) - \delta_{ij} \frac{|\overrightarrow{OX_{ij}}|}{|\overrightarrow{OX_i}|} \right)^2,$$

where

$$\overrightarrow{OX_{ij}} = -\frac{\overrightarrow{X_iX_j} \cdot \overrightarrow{OX_i}}{|\overrightarrow{X_iX_j}|^2} \overrightarrow{X_iX_j} + \overrightarrow{OX_i}, \qquad \text{and} \qquad \left(\frac{|\overrightarrow{OX_{ij}}|}{|\overrightarrow{OX_i}|} \right)^2 = 1 - \left(\frac{\overrightarrow{OX_i} \cdot \overrightarrow{X_iX_j}}{|\overrightarrow{OX_i}||\overrightarrow{X_iX_j}|} \right)^2.$$

The constant δ_{ij} is the penalty term, for example, which is coded 1 when the triangle X_iOX_j is adequate and 0 otherwise. The inadequate case is that angle X_iOX_j is smaller than angle X_iOX_{ij}. The factor λ_{ij} is the weight of the squared distance between data and model with respect to the ij-th element in the expanded ACP matrix, and the lift value between items i and j is a candidate.

3. Results

A math test containing 27 items was analyzed using the proposed method. The number of dimensions was set to be $Q = 3$ ($x_1 \in [-1,1]$, $x_2 \in [-1,1]$, $x_3 \in [0,1]$). However, the directions of the axes of the first and second dimensions are arbitrary. Therefore, the first and second coordinates of item 27 of which correct response rate is the lowest were set to be 0 ($x_{27,1}$=0) and larger than 0 ($x_{27,2}$>0), respectively. In addition, selecting an item of which conditional probability given that item 27 was correctly answered, the second coordinate of the item (item 11) was constrained to be larger than 0. Furthermore,

$$\lambda_{ij} = \begin{cases} 1 & \text{if } i,j \le n \\ n-1 & \text{otherwise} \end{cases}.$$

The genetic algorithm was used to optimize the stress function.

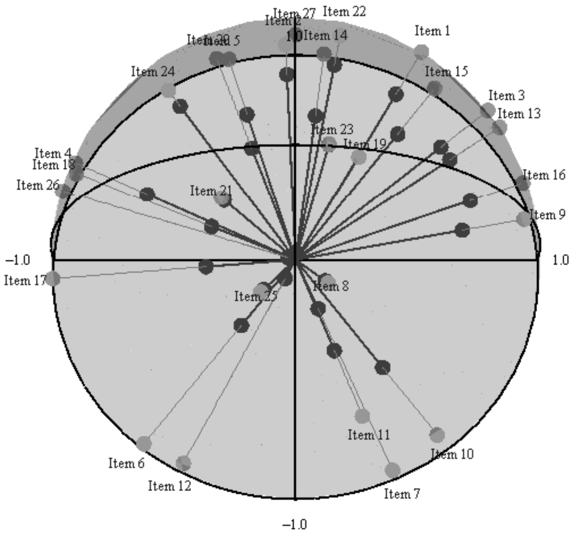

Fig. 2: Radial Plot

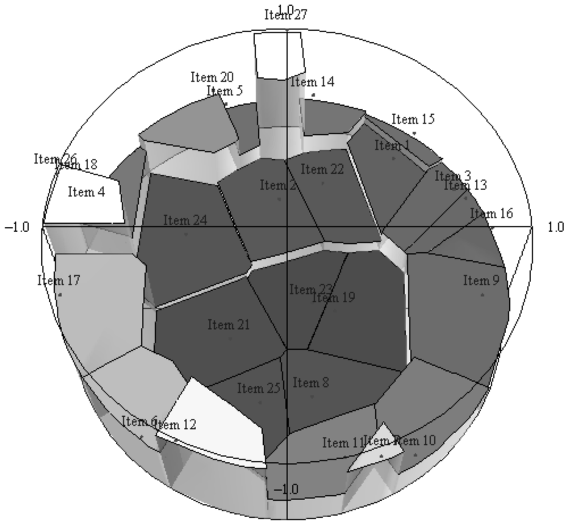

Fig. 3: Topographic Plot

Figure 2 is the radial plot obtained by the analysis. The plotted black points are the estimated coordinates of the items. All the black points are located inside the hemisphere. The black line segment of each item represents their easiness. The rays of easier items generally have a tendency to cluster around the top and more difficult items are inclined to lie horizontally with respect to the XY-plane. In addition, each gray point is located on the hemispherical surface and is an extension of the black line segment.

Figure 3 shows the topographic plot of the coordinates. The XY-coordinate of each item in the plot is identical to the XY-coordinate of the corresponding gray point in the radial plot, and the Z-coordinate in the topographic plot is the length of the gray line segment in the radial plot. In addition, the items are segmented by using Voronoi partitioning. In the topographic plot, more difficult items are more marginally located, lifted higher and colored lighter. Compared with the radial plot, the topographic plot makes it easier to understand the inter-item dependency structure.

Figure 4 shows the “skill mastery” maps of six examinees. The Voronoi cells of items that the examinee correctly answered are colored in the map of each examinee. Each map is an ability profile of the examinee, which items he/she correctly or incorrectly answered. The upper-left map is colored only around the center, so this examinee’s achievement level can be said to be low. The upper-center map indicates the examinee had a moderate level of achievement; he/she can be said to have mastered skills to be denoted by the upper-right region of the circle, which mainly consist of the items on two-dimensional vector diagrams. The colored cells are scattered in the lower-left map. It is likely that this examinee did not study systematically. The lower-center map indicates a rare occurrence. This examinee correctly answered comparatively difficult items but failed easier items. Such an examinee may as well repeatedly study the basics.

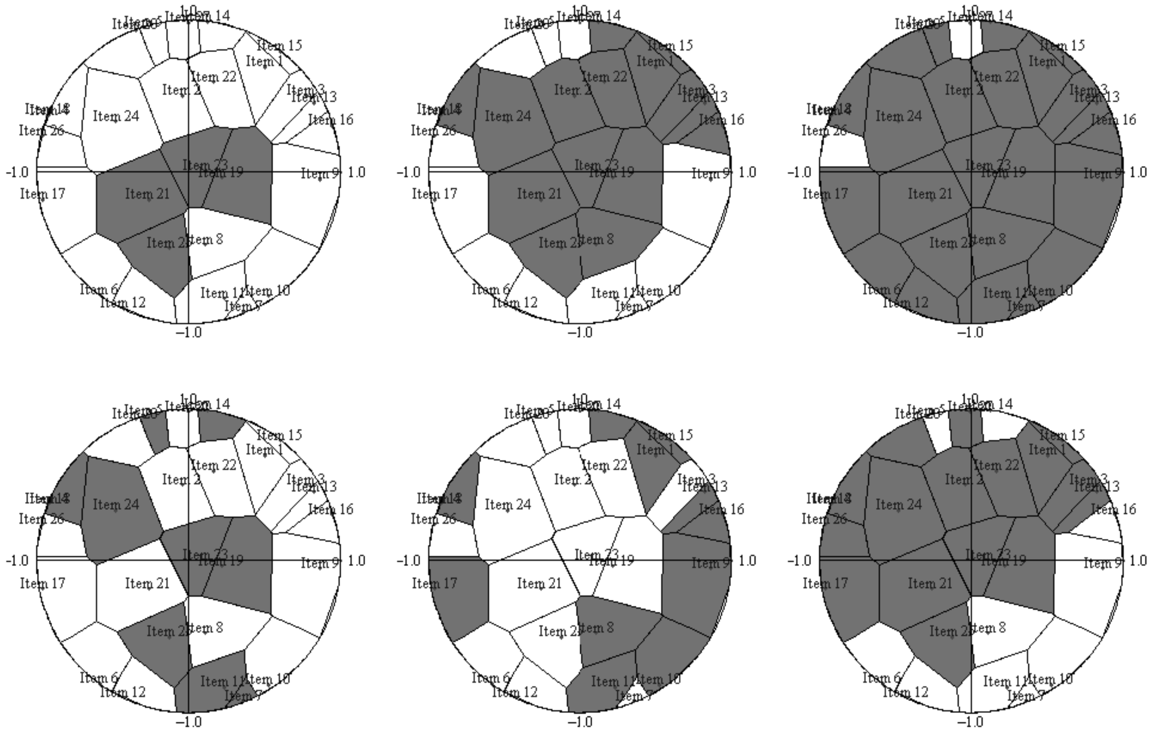

Fig. 4: Skill Mastery Map Examples

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