# Asymmetric von Mises Scaling

## Asymmetric multidimensional scaling using directional distribution

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### 1. Introduction

Asymmetric multidimensional scaling (AMDS) techniques are useful methods for analyzing an asymmetric data matrix. In this study, we propose an AMDS method, asymmetric von Mises scaling (AMISESCAL), in which von Mises distribution is used in directional statistics (Mardia and Jupp, 2000) to express asymmetric relationships among data objects.

The probability density function of a von Mises distribution has two parameters: the mean direction denoted by $\mu$ and the concentration parameter denoted by $\kappa$. It is given by

$$f(\theta|\mu,\kappa) = \frac{1}{2\pi I_0(\kappa)} \exp\{\kappa\cos(\theta-\mu)\} \quad \left(I_0(\kappa) = \frac{1}{2\pi}\int_0^{2\pi} exp\,\{\kappa\cos\theta\}d\theta\right).$$

Figure 1 shows four von Mises distributions with different parameters. In addition, Figure 2 represents the four distributions on the unit circle.

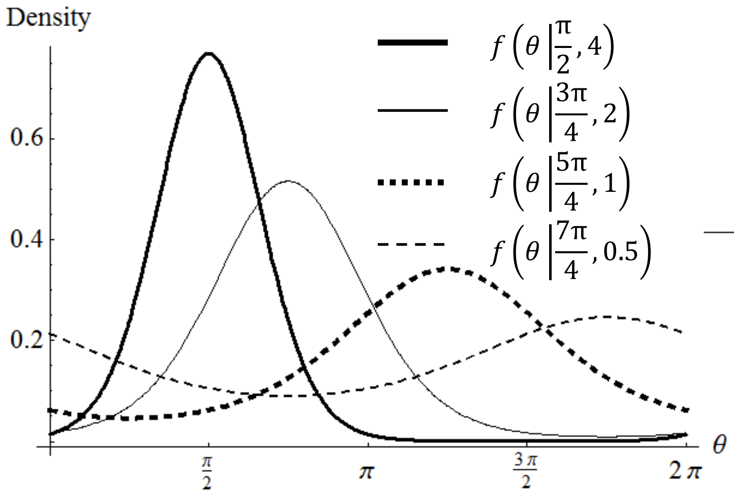


Figure 1: Von Mises Distribution

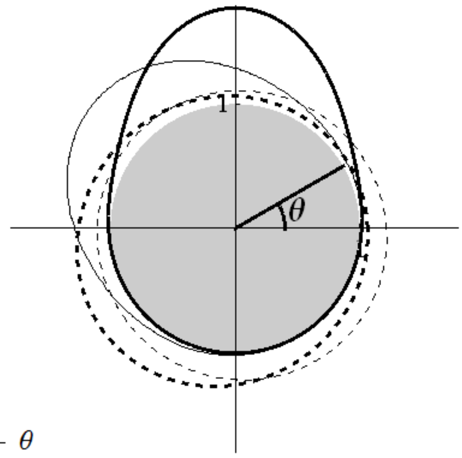


Figure 2: Von Mises Dist. on Unit Circle

### 2. Model

Let $N$ be the sample size and $\boldsymbol{G} = \{g_{ij}\}$ ($N\times N$) be the interpersonal distance data, where $g_{ij}$ (>0) represents the psychological proximity of person $i$ to person $j$; the smaller the value of $g_{ij}$, the more favorable is the feeling that person $i$ has for person $j$. In general, $g_{ij}$ is not always equal to $g_{ji}$; hence, $\boldsymbol{G}$ is obtained as an asymmetric square matrix. In addition, each diagonal element of the matrix represents the psychological proximity to oneself. Therefore, all the diagonal elements in $\boldsymbol{G}$ are set to 0.

In AMISESCAL, the skew-symmetry between $g_{ij}$ and $g_{ji}$ expressed in a multidimensional model space is explained by Figure 3. Let us assume that the coordinates of persons $i$ and $j$ are $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, and that they are located in the multidimensional model space. Let us also assume that the psychological proximity of person $i$ to person $j$ is smaller than that of person $j$ to person $i$. In other words, the preference value evaluated by person $i$ for person $j$ is larger than that evaluated by person $j$ for person $i$.

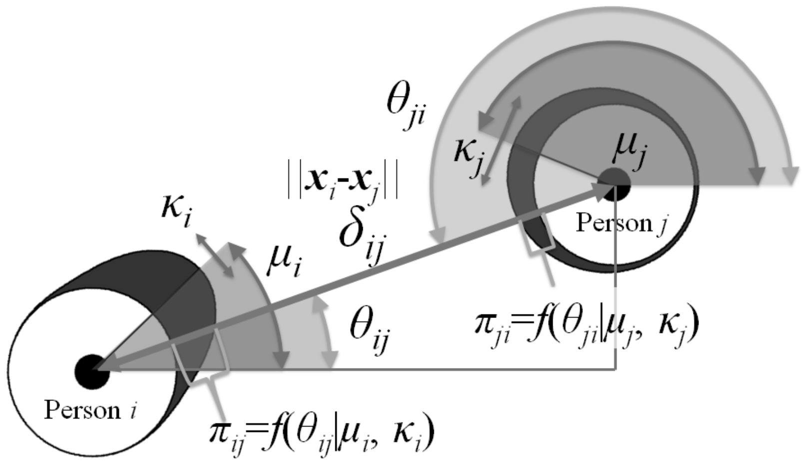


Figure 4: Skew-Symmetry of Coordinates of Persons $i$ and $j$ in AMISESCAL

In the figure, $\delta_{ij}$ $(=\delta_{ji})$ is the Euclidian distance between the coordinates $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. In addition, each person has two parameters, mean direction and concentration parameter, of one's own von Mises distribution, where $\mu_i$ represents person $i$'s "mean direction of his/her one-sided love" and $\kappa_i$ represents his/her "intensity of one-sided love." Furthermore, the direction angle from the coordinate of person $i$ to that of person $j$ is denoted by $\theta_{ij}$, and $\pi_{ij} = f(\theta_{ij}|\mu_i, \kappa_i)$ (s.t. $\pi_{ij} <$ 1.0) indicates the intensity of the feeling that person $i$ has for person $j$.

Then, the data $g_{ij}$ and $g_{ji}$ are modeled as $\xi_{ij} = (1\text{-}\pi_{ij})\delta_{ij}$ and $\xi_{ji} = (1\text{-}\pi_{ji})\delta_{ij}$, respectively, in the multidimensional model space, where $(1\text{-}\pi_{ij})$ can be understood as the scale reducing ratio that reflects the strength/weakness of person $i$'s feeling for person $j$. In Figure 4, $\pi_{ij}>\pi_{ji}$ indicates that the scale reducing ratio of person $i$ to person $j$ is smaller than that of person $j$ to person $i$.

Finally, the stress function to be optimized can be formulated as

$$S(\boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{\kappa}) = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(g_{ij} - \xi_{ij}\right)^2,$$

where $\boldsymbol{X}$ is the coordinate matrix, $\boldsymbol{\mu} = \{\mu_i\}$ $(N\times 1)$ is the mean direction parameter vector, and $\boldsymbol{\kappa} = \{\kappa_i\}$ $(N\times 1)$ is the concentration parameter vector to be estimated.

References


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